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Automata Theory Based on Quantum Logic: Recognizability and Accessibility

Jun Liu · Zhi-wen Mo

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Abstract Inspired by Ying's work on automata theory based on quantum logic and classical automata theory, we introduce the concepts of reversal, accessible, coaccessible and complete part of finite state automata based on quantum logic. Some properties of them are discussed. More importantly we investigate the recognizability and accessibility properties of these types on the framework of quantum logic by employing the approach of semantic analysis.

Keywords Quantum logic · Quantum automata

1 Introduction

Quantum logic was introduced by Birkhoff and Von Neumann [1] as the logic of quantum mechanics, and it stemmed from Von Neumann's Hilbert space formalization of quantum mechanics in which the behavior of quantum mechanical system is described by a closed space of Hilbert space. By noting that the set of all closed subspace of a Hilbert space consists of an orthomodular lattice, it was suggested that orthomodular lattices would be used as the algebraic version of the logic of quantum mechanics, just like Boolean algebras acting as an algebraic counterpart of classical logic. The major difference between Boolean logic and quantum logic is that the latter does not enjoy distributivity in general. Usually, orthomodular lattices are thought of as quantum logic [12]. Nowadays, with ever-increasing interest in quantum computing, different kinds of quantum logics have been put forward [3, 19]. They are based on different principles, for example based on probability [11].

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J. Liu (✉) · Z.-w. Mo
College of Mathematics and Software Science, Sichuan Normal University, Chengdu, Sichuan 610066, China
e-mail: liujun8476@163.com

Z.-w. Mo
e-mail: mozhiwen@263.net

Automata theory based on quantum logic has attracted attention in the academic community [4–7, 9, 13, 14]. A fundamental, yet particularly simple, model of quantum computation is a quantum finite state automata. Recently, Ying [20, 21] primarily and very significantly considered automata theory based on quantum logic (l-valued automata), in which quantum logic is understood as a logic whose truth-value set is an orthomodular lattice, and an element of an orthomodular lattice is assigned to each transition of an automaton and it is considered to be the truth value of the proposition describing the transition. This is a logical approach to quantum computation, and it should be treated as a further abstraction of mathematical models of quantum computation.

With this approach, Ying [20, 21] introduced the orthomodular lattice quantum predicate of recognizability, dealt with some operations on l-valued automata, and interestingly established corresponding pumping lemma. The author [22] reexamined carefully various properties of automata in the framework of quantum logic by employing the approach of semantic analysis. The closure properties of orthomodular l-valued regular languages are derived. The Kleene theorem about equivalence of regular expressions and finite state automata is generalized in to quantum logic. It is found that the universal validity of many properties (for example, the Kleene theorem, the equivalence of deterministic and nondeterministic automata) of automata depend heavily upon the distributivity of the underlying logic. This indicates that these properties do not universally hold in the l-valued automata. These negative results may help us to clarify the boundary between classical computation and quantum computation. Afterwards many researchers [2, 10, 15, 16, 18] initiated the further study of quantum finite automata introduces by Ying [20, 21].

This paper is further study on l-valued finite automata by employing the approach of semantic analysis. We introduce the concepts of reversal, accessible and complete based on the quantum logic. Some properties of them are discussed. More importantly we investigate the recognizability and accessibility properties of these types on the framework of quantum logic by employing the approach of semantic analysis.

2 Preliminaries

In this section, we review some basic notions and terminology needed in the subsequent sections and to fix notations. For more details, we refer to [8, 22]. The set of truth values of a quantum logic will be taken to be an orthomodular lattice. So we first recall the notion of orthomodular lattice.

An orthomodular lattice is a 7-tuple $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1\rangle$, where:

(1) $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1\rangle$ is a complete lattice, 0,1 are the least and greatest elements of L, respectively; $\leq$ is the partial ordering in L; $\wedge M$ and $\vee M$ stand for the greatest lower bound and the least upper bound of M, respectively.

(2) $\perp$ is a unary operation on L, called orthocomplement, and required to satisfy the following conditions, for any $a, b \in L$:
  - (i) $a \wedge a^{\perp} = 0, a \vee a^{\perp} = 1$;
  - (ii) $a^{\perp\perp} = a$;
  - (iii) $a \leq b$ implies $b^{\perp} \leq a^{\perp}$;
  - (iv) $a \wedge (a^{\perp} \vee (a \wedge b)) \leq b$.

A quantum logic is a complete orthomodular l-valued logic. We use the Sasaki arrow as the implication operation operator. The Sasaki arrow is defined as follows: for any $a, b \in L$,

$$a \to b \stackrel{\text{def}}{=} a^{\perp} \vee (a \wedge b)$$

(3) $a \le b$ iff $a \to b = 1$.

The bi-implication operator corresponding to the Sasaki arrow is defined as follows:

(4) $a \leftrightarrow b \overset{\text{def}}{=} (a \to b) \wedge (b \to a)$ for any $a, b \in L$.

Let $l = \langle L, \le, \wedge, \vee, \perp, 0, 1\rangle$ be a complete orthomodular lattice and $\to$ the Sasaki arrow. The syntax of l-valued logic is similar to that of classical first-order logic. We have three primitive connectives $\neg$ (negation), $\wedge$ (conjunction), and $\to$ (implication) and a primitive quantifier $\forall$ (universal quantifier). The connectives $\vee$ (disjunction), $\leftrightarrow$ (bi-implication), and the existential quantifier $\exists$ are defined in terms of $\neg, \wedge, \to$ and $\forall$ in the usual way.

In addition, the semantics of l-valued logic is given by interpreting the connectives $\neg$, $\wedge$, and $\to$ as the operations $\perp$, $\wedge$, and $\to$, respectively, on L and by interpreting the quantifier $\forall$ as the greatest lower bound in L. In addition, the truth value of set-theoretical formula $x \in A$ is $\lceil x \in A \rceil = A(x)$. It is worth indicating that in this paper the set A and its characteristic function are identified. In the l-valued logic, 1 is the unique designated truth value. In other words, a formula φ is valid iff its truth valid $\lceil \varphi \rceil$ is 1.

Secondly, we recall the concept of quantum automaton. Let $l = \langle L, \le, \wedge, \vee, \perp, 0, 1\rangle$ be an orthomodular lattice, and let Σ be a finite alphabet whose elements are called labels. Then an l-valued(quantum) automaton over Σ is a quadruple $\Re = \langle Q, I, T, \mu\rangle$, where:

(i) Q is a finite set whose elements are called states;
(ii) I is an l-valued subset of Q; that is, a mapping from Q into L. For each $q \in Q$, $I(q)$ indicates the truth value of the proposition that q is an initial state;
(iii) T is also an l-valued subset of Q, and for every $q \in Q$, $T(q)$ express the truth value of the proposition that q is terminal;
(iv) μ is an l-valued subset of $Q \times \Sigma \times Q$, i.e., a mapping from $Q \times \Sigma \times Q$ into L and it is called the l-valued (quantum) transition relation of $\Re$. Intuitively, for any $p, q \in Q$ and $\sigma \in \Sigma$, $\mu(p, \sigma, q)$ indicates the truth value of the proposition that input σ causes state p to become p.

We write $\mathbf{A}(\Sigma, l)$ for the (proper) class of all l-valued automata over Σ.

In classical automata theory, a path $c = q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k$ is said to be successful if $q_0 \in I$ and $q_k \in T$. The behavior of an automaton is the set of labels of all successful paths in $\Re$.

We set

$$T(Q, \Sigma) = (Q \times \Sigma)^* \times Q = \bigcup_{n=0}^{\infty} [(Q \times \Sigma)^n \times Q]$$

For any $c = q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k \in T(Q, \Sigma)$, k is the length of c and denoted by $|c|$, q_0 is the beginning of c and denoted by $b(c)$, q_k is the end of c and denoted by $e(c)$, and sequence $s = \sigma_1 \ldots \sigma_k$ is called the label of c and denoted by $lb(c)$.

Now, we review the key notion—recognizability for l-valued automata.

An l-valued (unary) predicate $path_{\Re}$ on $T(Q, \Sigma)$ is defined as $path_{\Re} \in L^{T(Q,\Sigma)}$ [the set of all mappings from $T(Q, \Sigma)$ into L]: for every $q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k \in T(Q, \Sigma)$,

$$path_{\Re}(q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k) \overset{\text{def}}{=} \bigwedge_{i=0}^{k-1} [(q_i, \sigma_{i+1}, q_{i+1}) \in \mu]$$

Intuitively, the truth value of the proposition that $q_0\sigma_1 q_1 \dots q_{k-1}\sigma_k q_k$ is a path in $\Re$ is

$$\lceil path_{\Re}(q_0\sigma_1 q_1 \dots q_{k-1}\sigma_k q_k)\rceil \overset{\text{def}}{=} \bigwedge_{i=0}^{k-1} \mu(q_i, \sigma_{i+1}, q_{i+1})$$

An l-valued automaton over Σ determines an l-valued (unary) predicate $rec_{\Re}$ on $\Sigma^* = \{\varepsilon\} \cup \bigcup_{k=1}^{\infty} \Sigma^k$ is defined as follows: for every $s \in \Sigma^*$,

$$rec_{\Re}(s) \overset{\text{def}}{=} (\exists c \in T(Q, \Sigma))(b(c) \in I \wedge e(c) \in T \wedge lb(c) = s \wedge path_{\Re}(c))$$

Intuitively, the truth value of the proposition that s is recognizable by $\Re$ is

$$\begin{aligned}\lceil rec_{\Re}(s)\rceil = \bigvee\{&I(b(c)) \wedge T(e(c))\\ &\wedge \lceil path_{\Re}(c)\rceil : c \in T(Q, \Sigma), b(c) \in I, e(c) \in T, lb(c) = s\}\end{aligned}$$

3 Reversal l-Valued Automata

Definition 3.1 The mapping $\rho : \Sigma^* \to \Sigma^*$ is called *reversal*, if $\rho(\varepsilon_\Sigma) = \varepsilon_\Sigma$, $\rho(\sigma) = \sigma$, $\rho(x\sigma) = \rho(\sigma)\rho(x)$, $\rho(\rho(x)) = x$, $\forall \sigma \in \Sigma$ and $x \in \Sigma^*$.

It can be easily proved inductively that $\rho(xy) = \rho(y)\rho(x), \forall x, y \in \Sigma^*$. The reversal mapping ρ defines the l-valued set $\mu^\rho : Q \times \Sigma \times Q \to L$ such that $\mu^\rho(p, \sigma, q) = \mu(q, \rho(\sigma), p)$.

Let $\Re = \langle Q, I, T, \mu\rangle \in \mathbf{A}(\Sigma, l)$ and $\rho : \Sigma^* \to \Sigma^*$ be the reversal mapping. Then $\Re^\rho = \langle Q, T, I, \mu^\rho\rangle \in \mathbf{A}(\Sigma, l)$ and is called the *reversal* of $\Re$.

Proposition 3.2 *The following two statements are equivalent*:

(i) *L satisfies the distributivity*: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, *for any* $a, b, c \in L$.
(ii) $\overset{l}{\models} \mu^\rho(p, xy, q) \leftrightarrow (\exists r \in Q)(\mu(q, \rho(y), r) \wedge (\mu(r, \rho(x), p)))$.

Proof (i) $\Rightarrow$ (ii) It suffices to show that $\mu^\rho(p, xy, q) = \bigvee\{\mu(q, \rho(y), r) \wedge \mu(r, \rho(x), p) : r \in Q, \forall p, q \in Q\}$. We prove the result by induction on $|y| = k$, where $|y|$ denotes the length of word y. If $k = 0$, then $y = \varepsilon$, it is clear by the definitions of μ^ρ and μ. Suppose the result is true for all $y \in \Sigma^*$ such that $|y| = k - 1, k > 0$. Let $y = \sigma_1 \dots \sigma_k \in \Sigma^*, \sigma_1, \dots, \sigma_k \in \Sigma$. Then from the definitions of μ^ρ, μ and the condition (i), we have

$$\begin{aligned}
&\mu^\rho(p, xy, q)\\
&\quad = \bigvee\{\mu^\rho(p, x\sigma_1 \dots \sigma_{k-1}, r) \wedge \mu^\rho(r, \sigma_k, q) : r \in Q\}\\
&\quad = \bigvee\{\vee\{\mu(r', \rho(x), p) \wedge \mu(r, \rho(\sigma_1 \dots \sigma_{k-1}), r') : r' \in Q\} \wedge \mu(q, \rho(\sigma_k), r) : r \in Q\}\\
&\quad = \bigvee\{\mu(r', \rho(x), p) \wedge \mu(r, \rho(\sigma_1 \dots \sigma_{k-1}), r') \wedge \mu(q, \rho(\sigma_k), r) : r', r \in Q\}\\
&\quad = \bigvee\{\mu(r', \rho(x), p) \wedge \vee\{\mu(r, \rho(\sigma_1 \dots \sigma_{k-1}), r') \wedge \mu(q, \rho(\sigma_k), r) : r \in Q\} : r' \in Q\}\\
&\quad = \bigvee\{\mu(r', \rho(x), p) \wedge \mu(q, \rho(\sigma_1 \dots \sigma_k), r') : r' \in Q\}\\
&\quad = \bigvee\{\mu(q, \rho(y), r') \wedge \mu(r', \rho(x), p) : r' \in Q\}
\end{aligned}$$

(ii) ⇒ (i) Given $a, b, c \in L$, we prove l enjoys distributivity; that is, for any $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. We take $\Re = \langle Q, I, T, \mu\rangle$, $\Re^\rho = \langle Q, T, I, \mu^\rho\rangle \in \mathbf{A}(\Sigma, l)$, where $Q = p_0, p_1, p_2, p_3, p_4$, $I = p_0$, $T = p_4$ and $\mu(p_0, \rho(\sigma_3), p_1) = a$, $\mu(p_1, \rho(\sigma_2), p_2) = \mu(p_1, \rho(\sigma_2), p_3) = 1$, $\mu(p_2, \rho(\sigma_1), p_4) = b$, $\mu(p_3, \rho(\sigma_1), p_4) = c$ for some $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$, and μ takes 0 for other arguments. Let $x = \sigma_1$ and $y = \sigma_2\sigma_3$. Then it follows that

$$\begin{aligned}
\mu^\rho(p_4, xy, p_0) &= \bigvee\{\mu(p_0, \rho(\sigma_2\sigma_3), r') \wedge \mu(r', \rho(\sigma_1), p_4) : r' \in Q\} \\
&= (\mu(p_0, \rho(\sigma_3)\rho(\sigma_2), p_2) \wedge \mu(p_2, \rho(\sigma_1), p_4)) \\
&\quad \vee (\mu(p_0, \rho(\sigma_3)\rho(\sigma_2), p_3) \wedge \mu(p_3, \rho(\sigma_1), p_4)) \\
&= ((a \wedge 1) \wedge b) \vee ((a \wedge 1) \wedge c) \\
&= (a \wedge b) \vee (a \wedge c) \\
&\quad \vee \{\mu(p_0, \rho(\sigma_3), r') \wedge \mu(r', \rho(\sigma_2)\rho(\sigma_1), p_4) : r' \in Q\} \\
&= \mu(p_0, \sigma_3, p_1) \wedge \mu(p_1, \rho(\sigma_2)\rho(\sigma_1), p_4) \\
&= a \wedge \vee\{\mu(p_1, \rho(\sigma_2), r) \wedge \mu(r, \rho(\sigma_1), p_4) : r \in Q\} \\
&= a \wedge ((\mu(p_1, \rho(\sigma_2), p_2) \wedge \mu(p_2, \rho(\sigma_1), p_4)) \\
&\quad \vee (\mu(p_1, \rho(\sigma_2), p_3) \wedge \mu(p_3, \rho(\sigma_1), p_4)) \\
&= a \wedge ((1 \wedge b) \vee (1 \wedge c)) \\
&= a \wedge (b \vee c)
\end{aligned}$$

So $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. □

Proposition 3.3 *Let $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1\rangle$ be a complete orthomodular lattice. If the implication operator $\rightarrow$ satisfies that $a \leftrightarrow a = 1$ for any $a \in L$. Then for any $\Re \in \mathbf{A}(\Sigma, l)$ and for any $s \in \Sigma^*$,*

$$\overset{l}{\models} rec_{\Re^\rho}(s) \leftrightarrow rec_{\Re}(\rho(s))$$

Proof Let $s = \sigma_1 \ldots \sigma_k$. Then

$$\begin{aligned}
\lceil rec_{\Re}(\rho(s))\rceil &= \lceil rec_{\Re}(\rho(\sigma_k)\ldots\rho(\sigma_1))\rceil \\
&= \bigvee\left\{ I(q_k) \wedge T(q_0) \wedge \bigwedge_{i=0}^{k-1} \mu(q_{j+1}, \rho(\sigma_{j+1}), q_j) : q_0 \in T, q_k \in I, q_i \in Q, \right. \\
&\qquad \left. i = 1, \ldots, k-1 \right\} \\
&= \bigvee\left\{ T^\rho(q_k) \wedge I^\rho(q_0) \wedge \mu^\rho(q_{k-1}, \sigma_k, q_k) \wedge \ldots \wedge \mu^\rho(q_0, \sigma_1, q_1) : q_0 \in I^\rho, \right. \\
&\qquad \left. q_k \in T^\rho, q_i \in Q \right\}
\end{aligned}$$

$$= \bigvee \left\{ I^\rho(q_0) \wedge T^\rho(q_k) \wedge \bigwedge_{i=0}^{k-1} \mu^\rho(q_i, \sigma_{i+1}, q_{i+1}) : q_0 \in I^\rho, q_k \in T^\rho, q_i \in Q, \right.$$

$$\left. i = 1, \ldots, k-1 \right\}$$

$$= \lceil rec_{\Re^\rho}(s) \rceil \qquad \square$$

4 Accessible *l*-Valued Automata

Definition 4.1 Let $\Re \in \mathbf{A}(\Sigma, l)$. Then the l-valued accessible predicate $acce_\Re$ on Q is defined as $acce_\Re \in L^Q$: for every $q \in Q$,

$$acce_\Re(q) \stackrel{\text{def}}{=} (\exists s \in \Sigma^*)(\exists c \in T(Q, \Sigma)(b(c) \in I \wedge e(c) = q \wedge path_\Re(c))$$

Thus, the truth value of the proposition that q is accessible by $\Re$ is given by

$$\lceil acce_\Re(q) \rceil = \bigvee \{ I(b(c)) \wedge X(q) \wedge \lceil path_\Re(c) \rceil : c \in T(Q, \Sigma), e(c) = q, lb(c) \in \Sigma^* \}$$

Let $Q^a = \{q \in Q | \lceil acce_\Re(q) \rceil > 0\}$, $\mu^a = \mu|_{Q^a \times \Sigma \times Q^a}$, $I^a = I|_{Q^a}$, and $T^a = T|_{Q^a}$. Then $\Re^a = \langle Q^a, I^a, T^a, \mu^a \rangle \in \mathbf{A}(\Sigma, l)$ is called accessible part of $\Re$.

In classical automata theory, a word ω of Σ^* is said to be recognized if there exists a successful path with label ω, $i\omega \in T$. A state q is accessible if there exists ω in Σ^* such that $i\omega = q$. Now, in the l-valued automata theory, we compare with the two notions: recognizable predicate and accessible predicate. In the former, for every $s \in \Sigma^*$, once s is given, then s is fixed and $lb(c) = s$. Its truth value is determined by the different successful $paths_\Re(c)$ with $lb(c) = s, b(c) \in I, e(c) \in T$. However, in the latter, the different $paths_\Re(c)$ are not all successful. It only needs to satisfy $e(c) = q$. So $lb(c)(\in \Sigma^*)$ is variable, not a constant s. Its truth value is decided by the different $lb(c)(\in \Sigma^*)$ and $e(c) = q$.

The idea of accessibility has an obvious dual.

Definition 4.2 Let $\Re \in \mathbf{A}(\Sigma, l)$. Then the l-valued coaccessible predicate $coacce_\Re$ on Q is defined as $coacce_\Re \in L^Q$: for every $q \in Q$,

$$coacce_\Re(q) \stackrel{\text{def}}{=} (\exists s \in \Sigma^*)(\exists c \in T(Q, \Sigma)(b(c) = q \in I \wedge e(c) \in T \wedge path_\Re(c))$$

The truth value of the proposition that q is accessible by $\Re$ is given by

$$\lceil coacce_\Re(q) \rceil = \bigvee \{ X(q) \wedge T(e(c)) \wedge \lceil path_\Re(c) \rceil : c \in T(Q, \Sigma), b(c) = q, lb(c) \in \Sigma^* \}$$

Let $Q^b = \{q \in Q | \lceil coacce_\Re(q) \rceil > 0\}$, $\mu^b = \mu|_{Q^b \times \Sigma \times Q^b}$, $I^b = I|_{Q^b}$, and $T^b = T|_{Q^b}$. Then $\Re^b = \langle Q^b, I^b, T^b, \mu^b \rangle \in \mathbf{A}(\Sigma, l)$ is called coaccessible part of $\Re$.

Proposition 4.3 *Let $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1 \rangle$ be a complete orthomodular lattice, and let the implication operator $\to$ fulfill the property that $a \leftrightarrow a = 1$ for any $a \in L$. Then for any $\Re \in \mathbf{A}(\Sigma, l)$ and for any $s \in \Sigma^*$, it holds that*

$$\text{(i)} \stackrel{l}{\models} rec_{\Re^a}(s) \leftrightarrow rec_\Re(s), \qquad \text{(ii)} \stackrel{l}{\models} rec_{\Re^b}(s) \leftrightarrow rec_\Re(s)$$

Proof Let $s=\sigma_1\ldots\sigma_k$. Then

$$\begin{aligned}
\lceil rec_{\Re}(s)\rceil &= \bigvee\{I(q_0)\wedge T(q_k)\wedge \lceil path_{\Re}(c)\rceil : c\in T(Q,\Sigma),\ \text{and } lb(c)=s\}\\
&= \bigvee\left\{I^a(q_0)\wedge T^a(q_k)\wedge\bigwedge_{i=0}^{k-1}\mu^a(q_i,\sigma_{i+1},q_{i+1}) : q_0,\ldots,q_k\in Q^a\right\}\\
&\quad \vee\bigvee\left\{I^a(q_0)\wedge T(q_k)\wedge\bigwedge_{i=0}^{k-1}\mu(q_i,\sigma_{i+1},q_{i+1}) :\ \text{for some } q_i\in Q\backslash Q^a\right\}\\
&= \bigvee\left\{I^a(q_0)\wedge T^a(q_k)\wedge\bigwedge_{i=0}^{k-1}\mu^a(q_i,\sigma_{i+1},q_{i+1}) : q_0,\ldots,q_k\in Q^a\right\}\\
&= \lceil rec_{\Re^a}(s)\rceil
\end{aligned}$$

(ii) Similar to the proof of (i). □

Proposition 4.4 *Let $l=\langle L,\le,\wedge,\vee,\perp,0,1\rangle$ be a complete orthomodular lattice, and let the implication operator $\to$ fulfill the property that $a\leftrightarrow a=1$ for any $a\in L$. Then for any $q\in Q$, we have*

$$\overset{l}{\models} coacce_{\Re}(q)\leftrightarrow acce_{\Re^\rho}(q).$$

Proof For any $\Re=\langle Q,I,T,\mu\rangle\in\mathbf{A}(\Sigma,l)$, then $\Re^\rho=\langle Q,I^\rho,T^\rho,\mu^\rho\rangle$, and $I^\rho=T$, $T^\rho=I$, $\mu^\rho(q,\sigma,p)=\mu(p,\rho(\sigma),q)$. Now by using Definitions 4.1 and 4.2 we obtain

$$\begin{aligned}
\lceil coacce_{\Re}(q)\rceil &= \bigvee\{X(q)\wedge T(q_k)\wedge\lceil path_{\Re}(c)\rceil : q_k\in T, c\in T(Q,\Sigma),\\
&\qquad \text{and } b(c)=q,\ lb(c)\in\Sigma^*\}\\
&= \bigvee\left\{X(q)\wedge T(q_k)\wedge\bigwedge_{i=0}^{k-1}\mu(q_i,\sigma_{i+1},q_{i+1}) : q_k\in T,\right.\\
&\qquad \left. q_0=q,\ c\in T(Q,\Sigma),\ \text{and } b(c)=q,\ lb(c)\in\Sigma^*\right\}\\
&= \bigvee\left\{X(q)\wedge I^\rho(q_k)\wedge\bigwedge_{i=0}^{k-1}\mu^\rho(q_{i+1},\rho(\sigma_{i+1}),q_i) : q_k\in T,\right.\\
&\qquad \left. q_0=q,\ c\in T(Q,\Sigma),\ \text{and } b(c)=q, lb(c)\in\Sigma^*\right\}\\
&= \bigvee\{I^\rho(q_k)\wedge X(q)\wedge\mu^\rho(q_k,\rho(\sigma_1\ldots\sigma_k),q) : q_k\in I^\rho,\\
&\qquad q_0=q,\ c\in T(Q,\Sigma),\ \text{and } b(c)=q, lb(c)\in\Sigma^*\}\\
&= \lceil rec_{\Re^\rho}(q)\rceil
\end{aligned}$$

□

Definition 4.5 Let $\mathfrak{R} \in \mathbf{A}(\Sigma, l)$. Then $\mathfrak{R}$ is called trim, if it is both accessible and coaccessible. Let $Q^t = \{q \in Q | \lceil acce_{\mathfrak{R}}(q) \rceil \wedge \lceil coacce_{\mathfrak{R}}(q) \rceil > 0\}$. We denote $\mathfrak{R}^t = \langle Q^t, I^t, T^t, \mu^t \rangle$, $\mu^t = \mu|_{Q^t \times \Sigma \times Q^t}$, $I^t = I|_{Q^t}$, and $T^t = T|_{Q^t}$. Then $\mathfrak{R}^t$ is called trim part of $\mathfrak{R}$.

Proposition 4.6 *Let $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1 \rangle$ be a complete orthomodular lattice, and let the implication operator $\rightarrow$ fulfill the property that $a \leftrightarrow a = 1$ for any $a \in L$. Then for any $\mathfrak{R} \in \mathbf{A}(\Sigma, l)$ and for any $s \in \Sigma^*$, it holds that*

$$\overset{l}{\models} rec_{\mathfrak{R}^t}(s) \leftrightarrow rec_{\mathfrak{R}}(s)$$

Proof It is obvious by the Proposition 4.3. □

Definition 4.7 Let $f_1 : Q_1 \rightarrow L$, $f_2 : Q_2 \rightarrow L$

(1) $f_1 \wedge f_2 : Q_1 \times Q_2 \rightarrow L$ is defined by $f_1 \wedge f_2(q_1, q_2) = f_1(q_1) \wedge f_2(q_2)$

(2) If $Q_1 \cap Q_2 = \emptyset$, then $f_1 \vee f_2 : Q_1 \cup Q_2 \rightarrow L$ is defined by

$$f_1 \vee f_2(q) = \begin{cases} f_1(q), & \text{if } q \in Q_1 \\ f_2(q), & \text{if } q \in Q_2 \end{cases}$$

Definition 4.8 Let $\mathfrak{R} = \langle Q_1, I_1, T_1, \mu_1 \rangle$, and $\wp = \langle Q_2, I_2, T_2, \mu_2 \rangle \in \mathbf{A}(\Sigma, l)$ be two l-valued automata over Σ. Then their product $\mathfrak{R} \times \wp$ is defined to be $\mathfrak{I} = \langle Q_3, I_3, T_3, \mu_3 \rangle$, where:

(i) $Q_3 = Q_1 \times Q_2$;
(ii) $I_3 = I_1 \wedge I_2$;
(iii) $T_3 = T_1 \wedge T_2$;
(iv) $\mu_3 : Q_3 \times \Sigma \times Q_3 \rightarrow L$ and for any $p_1, q_1 \in Q_1$, $p_2, q_2 \in Q_2$, and $\sigma \in \Sigma$,

$$\mu_3((p_1, p_2), \sigma, (q_1, q_2)) = \mu_1(p_1, \sigma, q_1) \wedge \mu_2(p_2, \sigma, q_2)$$

Proposition 4.9 *Let $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1 \rangle$ be a complete orthomodular lattice*:

(1) *For any $\mathfrak{R}, \wp \in \mathbf{A}(\Sigma, l)$ and for any $(p, q) \in Q_1 \times Q_2$, we have*

$$\overset{l}{\models} acce_{\mathfrak{R} \times \wp}(p, q) \rightarrow acce_{\mathfrak{R}}(p) \wedge acce_{\wp}(q)$$

(2) *If L satisfies the distributivity*: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, *for any $a, b, c \in L$. Then for any $\mathfrak{R}, \wp \in \mathbf{A}(\Sigma, l)$, and for any $(p, q) \in Q_1 \times Q_2$,*

$$\overset{l}{\models} acce_{\mathfrak{R}}(p) \wedge acce_{\wp}(q) \leftrightarrow acce_{\mathfrak{R} \times \wp}(p, q)$$

Proof For any $(p, q) \in Q_1 \times Q_2$, then

$$\lceil acce_{\mathfrak{R} \times \wp}(p, q) \rceil$$
$$= \bigvee \left\{ (I_1 \wedge I_2)(p_{10}, q_{20}) \wedge (X_1 \wedge X_2)(p, q) \wedge \bigwedge_{i=0}^{k-1} \mu_3((p_{1i}, q_{2i}), \sigma_{i+1}, (p_{1(i+1)}, q_{2(i+1)})) : \right.$$

$$p_{1i} \in Q_1, q_{2i} \in Q_2, lb(c) \in \Sigma^* \Bigg\}$$

$$= \bigvee \Bigg\{ I_1(p_{10}) \wedge I_2(q_{20}) \wedge X_1(p) \wedge X_2(q) \bigwedge_{i=0}^{k-1} \mu_1(p_{1i}, \sigma_{i+1}, p_{1(i+1)})$$

$$\wedge \bigwedge_{i=0}^{k-1} \mu_2(q_{2i}, \sigma_{i+1}, q_{2(i+1)}) : p_{1i} \in Q_1, q_{2i} \in Q_2 \Bigg\}$$

$$= \bigvee \Bigg\{ I_1(p_{10}) \wedge X_1(p) \wedge \bigwedge_{i=0}^{k-1} \mu_1(p_{1i}, \sigma_{i+1}, q_{1(i+1)}) : p_{1i} \in Q_1, i = 0, \ldots, k,$$

$$\text{and } p_{1k} = p, lb(c) \in \Sigma^* \Bigg\}$$

$$\vee \bigvee \Bigg\{ I_2(q_{20}) \wedge X_2(q) \wedge \bigwedge_{i=0}^{k-1} \mu_2(q_{2i}, \sigma_{i+1}, q_{2(i+1)}) : q_{2i} \in Q_2, i = 0, \ldots k,$$

$$\text{and } q_{2k} = q, lb(c) \in \Sigma^* \Bigg\}$$

$$\lceil acce_{\Re}(p) \wedge acce_{\wp}(q) \rceil = \Bigg[\bigvee \Bigg\{ I_1(p_0) \wedge X_1(p) \wedge \bigwedge_{i=0}^{k-1} \mu_1(p_i, \sigma_{i+1}, p_{i+1}) :$$

$$p_i \in Q_1, i = 0, \ldots k, \text{ and } p_k = p, lb(c) \in \Sigma^* \Bigg\} \Bigg]$$

$$\vee \Bigg[\bigvee \Bigg\{ I_2(q_0) \wedge X_2(q) \wedge \bigwedge_{i=0}^{k-1} \mu_2(q_i, \sigma_{i+1}, q_{i+1}) :$$

$$q_i \in Q_2, i = 0, \ldots k, \text{ and } q_k = q, lb(c) \in \Sigma^* \Bigg\} \Bigg]$$

It is clear that

$$\lceil acce_{\Re \times \wp}(p, q) \rceil \leq \lceil acce_{\Re}(p) \wedge acce_{\wp}(q) \rceil$$

If L satisfies the distributivity: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, from the above result, we can obtain

$$\lceil acce_{\Re}(p) \wedge acce_{\wp}(q) \rceil = \lceil acce_{\Re}(p) \rceil \wedge \lceil acce_{\wp}(q) \rceil$$
$$= \lceil acce_{\Re \times \wp}(p, q) \rceil \qquad \square$$

Definition 4.10 Let $\Re = \langle Q_1, I_1, T_1, \mu_1 \rangle$, and $\wp = \langle Q_2, I_2, T_2, \mu_2 \rangle \in \mathbf{A}(\Sigma, l)$ be two l-valued automata over Σ. We assume that $Q_1 \cap Q_2 = \emptyset$. Then the union $\Re \cup \wp$ is defined to be $\Im = \langle Q_3, I_3, T_3, \mu_3 \rangle$, where:

(i) $Q_3 = Q_1 \cup Q_2$;
(ii) $I_3 = I_1 \vee I_2$;
(iii) $T_3 = T_1 \vee T_2$;
(iv) $\mu_3 : Q_3 \times \Sigma \times Q_3 \to L$ is given as follows: for any $p, q \in Q_3$, and $\sigma \in \Sigma$,

$$\mu_3(p,\sigma,q) = \begin{cases} \mu_1(p,\sigma,q), & \text{if } p,q \in Q_1 \\ \mu_2(p,\sigma,q), & \text{if } p,q \in Q_2 \\ 0, & \text{otherwise} \end{cases}$$

Proposition 4.11 *Let $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1\rangle$ be a complete orthomodular lattice. If the implication operator $\to$ fulfill the property that $a \leftrightarrow a = 1$, for any $a \in L$. Then for any $\Re$, $\wp \in \mathbf{A}(\Sigma, l)$ and for any $q \in Q_1 \cup Q_2$, we have*

$$\overset{l}{\models} acce_{\Re\cup\wp}(q) \leftrightarrow acce_{\Re}(q) \vee acce_{\wp}(q)$$

Proof For any $q \in Q_1 \cup Q_2$, then

$$\begin{aligned}
&\lceil acce_{\Re\cup\wp}(q)\rceil \\
&= \bigvee\Bigg\{(I_1 \wedge I_2)(q_0) \wedge (X_1 \wedge X_2)(q) \wedge \bigwedge_{i=0}^{k-1} \mu_3(q_i, \sigma_{i+1}, q_{(i+1)}) : \\
&\quad q_i \in Q_1 \cup Q_2, i = 0, \ldots, k, q_k = q, lb(c) \in \Sigma^*\Bigg\} \\
&= \Bigg[\bigvee\Bigg\{(I_1 \wedge I_2)(q_0) \wedge (X_1 \wedge X_2)(q) \wedge \bigwedge_{i=0}^{k-1} \mu_3(q_i, \sigma_{i+1}, q_{(i+1)}) : \\
&\quad q_i \in Q_1, i = 0, \ldots, k, q_k = q, lb(c) \in \Sigma^*\Bigg\}\Bigg] \\
&\quad \vee \Bigg[\bigvee\Bigg\{(I_1 \wedge I_2)(q_0) \wedge (X_1 \wedge X_2)(q) \wedge \bigwedge_{i=0}^{k-1} \mu_3(q_i, \sigma_{i+1}, q_{(i+1)}) : \\
&\quad q_i \in Q_2, i = 0, \ldots, k, q_k = q, lb(c) \in \Sigma^*\Bigg\}\Bigg] \\
&\quad \vee \Bigg[\bigvee\Bigg\{(I_1 \wedge I_2)(q_0) \wedge (X_1 \wedge X_2)(q) \wedge \bigwedge_{i=0}^{k-1} \mu_3(q_i, \sigma_{i+1}, q_{(i+1)}) : \\
&\quad q_i \in Q_1 \cup Q_2, i = 0, \ldots, k, q_k = q, lb(c) \in \Sigma^* \\
&\quad \text{and there are } i,\ j \text{ such that } 0 \leq i,\ j \leq\ k \text{ and } q_i \in Q_1,\ q_j \in Q_2\Bigg\}\Bigg] \\
&= \Bigg[\bigvee\Bigg\{I_1(q_0) \wedge X_1(q) \wedge \bigwedge_{i=0}^{k-1} \mu_1(q_i, \sigma_{i+1}, q_{(i+1)}) : \\
&\quad q_i \in Q_1, i = 0, \ldots, k,\ q_k = q,\ lb(c) \in \Sigma^*\Bigg\}\Bigg]
\end{aligned}$$

$$\vee\left[\bigvee\left\{I_2(q_0)\wedge X_2(q)\wedge\bigwedge_{i=0}^{k-1}\mu_2(q_i,\sigma_{i+1},q_{(i+1)}):\right.\right.$$

$$\left.\left.q_i\in Q_2, i=0,\ldots,k,\ q_k=q,\ lb(c)\in\Sigma^*\right\}\right]$$

$$=\lceil acce_{\Re}(q)\vee acce_{\wp}(q)\rceil$$ □

5 Complete l-Valued Automata

Definition 5.1 Let $\Re\in\mathbf{A}(\Sigma,l)$. Then $\Re=\langle Q,I,T,\mu\rangle$, is called complete, if for each $(p,\sigma)\in Q\times\Sigma$, there exist $q\in Q$, such that $\mu(p,\sigma,q)>0$.

Definition 5.2 Let $\Re\in\mathbf{A}(\Sigma,l)$. Then $\Re^c=\langle Q^c,I^c,T^c,\mu^c\rangle$, is called a completion of $\Re$, if:

(1) $\Re^c$ is a complete l-valued finite automaton.
(2) $\Re$ is the submachine of $\Re^c$, i.e., $Q\subseteq Q^c$, $I^c|_Q=I$, $T^c|_Q=T$, $\mu^c|_{Q\times\Sigma\times Q}=\mu$.

Example 5.3 Let $\Re\in\mathbf{A}(\Sigma,l)$ be an l-valued finite automaton, which is incomplete. Consider $\Re'=\langle Q',I',T',\mu'\rangle$, where $Q'=Q\cup\{z\}$, $z\bar{\in}Q$ and

$$\mu'(p,\sigma,q)=\begin{cases}\mu(p,\sigma,q), & \text{if } p,q\in Q,\ \mu(p,\sigma,q)\neq 0\\ 1, & \text{if either } \mu(p,\sigma,r)=0,\ \forall r\in Q,\ \text{and } q=z \text{ or } p=q=z\\ 0, & \text{otherwise}\end{cases}$$

where $I':Q'\to L$ be such that $I'(q)=I(q)$, if $q\in Q$, and 0, if $q\bar{\in}Q$, $T':Q'\to L$ be such that $T'(q)=T(q)$, if $q\in Q$, and 0, if $q\bar{\in}Q$. Then $\Re'$ is the completion of $\Re$.

Proof $\forall(q,\sigma)\in Q'\times\Sigma$:

(i) $q\in Q$. If $\mu(p,\sigma,r)=0$, $\forall r\in Q$, then take $p=z$, such that $\mu'(q,\sigma,p)>0$. If not,we take $r'\in Q$, such that $\mu'(p,\sigma,r)=\mu(p,\sigma,r)>0$.
(ii) $q=z$. We only take $p=z$. Hence, $\forall(q,\sigma)\in Q'\times\Sigma$, $\exists p\in Q$ such that $\mu'(q,\sigma,p)>0$.
Obviously, $Q\subseteq Q'$, $I'|_Q=I$, $T'|_Q=T$, $\mu'|_{Q\times\Sigma\times Q}=\mu$. □

Note It is clear that R' is the smallest completion.

Proposition 5.4 *Let $l=\langle L,\le,\wedge,\vee,\perp,0,1\rangle$ be a complete orthomodular lattice, and let the implication operator $\to$ fulfill the property that $a\leftrightarrow a=1$ for any $a\in L$. Then for any $\Re,\Re'\in\mathbf{A}(\Sigma,l)$, $\Re'$ is defined as the Example 5.3, and for any $s\in\Sigma^*$, it holds that*

$$\overset{l}{\models} rec_{\Re'}(s)\leftrightarrow rec_{\Re}(s)$$

Proof Let $s=\sigma_1\ldots\sigma_k$. Then

$$\begin{aligned}\lceil rec_{\Re'}(s)\rceil &= \lceil rec_{\Re'}(\sigma_1\ldots\sigma_k)\rceil\\ &=\bigvee\{I'(p_0)\wedge T'(p_k)\wedge\mu'(p_0,\sigma_1,p_1)\wedge\ldots\wedge\mu'(p_{k-1},\sigma_k,p_k):\\ &\quad p_0,\ldots,p_k\in Q'\}\end{aligned}$$

$$= \bigvee\{I(p_0) \wedge T(p_k) \wedge \mu(p_0,\sigma_1,p_1) \wedge \ldots \wedge \mu(p_{k-1},\sigma_k,p_k) : p_0,\ldots,p_k \in Q\}$$
$$\vee \bigvee\{I(p_0) \wedge T(p_k) \wedge \mu'(p_0,\sigma_1,p_1) \wedge \ldots \wedge \mu'(p_{k-1},\sigma_k,p_k) :$$
$$p_0, p_k \in Q, \text{ for some } p_i \in Q'\backslash Q\}$$

By the definition of μ', we obtain

$$\lceil rec_{\Re'}(s)\rceil = \bigvee\{I(p_0) \wedge T(p_k) \wedge \mu(p_0,\sigma_1,p_1) \wedge \ldots \wedge \mu(p_{k-1},\sigma_k,p_k) : p_0,\ldots,p_k \in Q\}$$
$$= \lceil rec_{\Re}(s)\rceil$$

□

Proposition 5.5 *Let $\Re \in \mathbf{A}(\Sigma, l)$ be an l-valued finite automaton. L satisfies the distributivity: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for any $a, b, c \in L$:*

(i) *If $\Re$ is a complete l-valued finite automaton, then so is $\Re^a$.*
(ii) *If $\Re$ is an accessible l-valued finite automaton, then so is $\Re^c$.*

Proof (i) Let $\Re = \langle Q, I, T, \mu\rangle$, $\Re^a = \langle Q^a, I^a, T^a, \mu^a\rangle$. For $(p,\sigma) \in Q^a \times \Sigma \subseteq Q \times \Sigma$, since $\Re$ is complete, there exists $q \in Q$ such that $\mu(p,\sigma,q) > 0$. Again $p \in Q^a$, then $\lceil acce_R(p)\rceil > 0$, i.e., $\exists q_0 \in Q$ and $s \in \Sigma^*$ such that $I(q_0) \wedge \mu(q_0,s,p) > 0$. Therefore, $I(q_0) > 0$ and $\mu(q_0,s,p) > 0$. So $\mu(q_0,s,p) \wedge \mu(p,\sigma,q) > 0$.

L satisfies the distributivity, hence $\mu(q_0,s\sigma,p) = \bigvee\{\mu(q_0,s,t) \wedge \mu(t,\sigma,q) | t \in Q\} > 0$. Thus $I(q_0) \wedge \mu(q_0,s\sigma,q) > 0$. Hence $q \in Q^a$, $\mu^a(p,\sigma,q) = \mu(p,\sigma,q)|_{Q^a \times \Sigma \times Q^a}$, then $\Re^a$ is complete.

(ii) Let $\Re$ be accessible. Therefore $Q = Q^a = \{q | \lceil acce_{\Re}(q)\rceil > 0\}$. Let $q^c \in Q^c$. If $q^c \in Q$, then we are done.

Since $\Re$ is incomplete, there exists $(p_0,\sigma_0) \in Q \times \Sigma$ such that $\mu(p_0,\sigma_0,t) = 0, \forall t \in Q$. But then $\mu(p_0,\sigma_0,q^c) = 1 > 0$. Again $\Re$ is accessible and $p_0 \in Q = Q^a$, there exist $r \in Q$ and $x_0 \in \Sigma^*$ such that $I(r) \wedge \mu(r,x_0,p_0) > 0$. Therefore $I(r) > 0$ and $\mu(r,x_0,p_0) > 0$. By the distributivity of L, so $\mu^c(r,x_0\sigma_0,q^c) = \bigvee\{\mu^c(r,x_0,k) \wedge \mu^c(k,x_0,q^c) | k \in Q\} > \mu^c(r,x_0,p_0) \wedge \mu^c(p_0,x_0,q^c) = \mu^c(r,x_0,p_0) = \mu(r,x_0,p_0) > 0$.

Thus $I(r) \wedge \mu^c(r,x_0\sigma_0,q^c) > 0$, i.e. q is accessible. Hence $\Re^c$ is accessible. □

Let $\Re = \langle Q, I, T, \mu\rangle$, be an l-valued automaton, $\wp(Q) = \{P | P \subseteq Q\}$. Define $\mu^d : \wp(Q) \times \Sigma \times \wp(Q) \to L$ by

$$\mu^d(P,\sigma,R) = \begin{cases} 1, & \text{if } P = \emptyset \text{ or } R = \emptyset \\ \bigvee\{\mu(p,\sigma,q) | p \in P, q \in R\}, & \text{otherwise} \end{cases}$$

$I^d : \wp(Q) \to L$ by

$$I^d(p) = \begin{cases} \bigwedge\{I(q) | p \in P\}, & \text{if } P \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and $T^d : \wp(Q) \to L$ by

$$T^d(p) = \begin{cases} \bigwedge\{T(p) | p \in P\}, & \text{if} P \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Then $\Re^d = \langle \wp(Q), I^d, T^d, \mu^d\rangle$ is a complete l-valued automaton.

Proposition 5.6 *Let $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1\rangle$, and let $\rightarrow$ be an implication operator satisfying the Birkhoff–von Neumann requirement. For $\Re^d = \langle \wp(Q), I^d, T^d, \mu^d\rangle \in \mathbf{A}(\Sigma, l)$, we have*

$$\overset{l}{\models} rec_{\Re}(s) \rightarrow rec_{\Re^d}(s)$$

Proof Let $s \in \Sigma^*$. Then

$$\begin{aligned}
&\lceil rec_{\Re^d}(s)\rceil \\
&\quad = \bigvee\{I^d(P) \wedge T^d(R) \wedge \mu^d(P, s, R) \mid P, R \in \wp(Q)\} \\
&\quad = \bigvee\Big\{\bigwedge\{I(p) \mid p \in P\} \wedge \bigwedge\{T(r) \mid r \in R\} \\
&\qquad \wedge \bigvee\{\mu(t, s, \nu) \mid t \in P, \nu \in R\} \mid P, R \in \wp(Q)\Big\} \\
&\quad = \bigvee\Big\{\bigwedge\{I(p) \wedge T(r) \mid p \in P, r \in R\} \\
&\qquad \wedge \bigvee\{\mu(t, s, \nu) \mid t \in P, \nu \in R\} \mid P, R \in \wp(Q)\Big\} \\
&\quad \geq \bigvee\Big\{\bigwedge\{I(p) \wedge T(r) \mid p \in P, r \in R\} \wedge \{\mu(t, s, \nu) \mid t \in P, \nu \in R\} \mid P, R \in \wp(Q)\Big\} \\
&\quad = \bigvee\{I(p) \wedge T(r) \wedge \mu(t, s, \nu) \mid p \in P,\ r \in R; t \in P,\ \nu \in R; P, R \in \wp(Q)\} \\
&\quad = \bigvee\{I(p) \wedge T(r) \wedge \mu(p, s, r) \mid p \in P, r \in R\} \\
&\qquad \vee \bigvee\{I(p) \wedge T(r) \wedge \mu(t, s, \nu) : t \neq p \text{ or } \nu \neq r\} \\
&\quad \geq \lceil rec_{\Re}(s)\rceil
\end{aligned}$$

□

6 Conclusion

In classical automata theory, given an automaton $\Re$ recognizing a set of words of the most general sort, we can guarantee to produce a deterministic and accessible automaton recognizing the set. If $\Re$ is non-deterministic we can determine it; If $\Re$ is not trim we can trim it; If $\Re$ is incomplete we can complete and we can do all of these without changing the recognizing set. This paper is devoted to an account of the basic properties of automata. We try to purse more properties of quantum recognizability and accessibility, but we find that the proof of some properties of automata by the semantic analysis method requires an essential application of the distributivity for the lattice of truth values of the underlying logic.

With the close relationship between automata theory and the theory of formal grammars, Qiu [15, 17] and Cheng and Wang [2] dealt with the equivalence relation between quantum automata and quantum grammars in the framework of quantum logic. We will consider the relation between Boolean logic and quantum logic and compare some basic properties in the different framework in subsequent work.

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